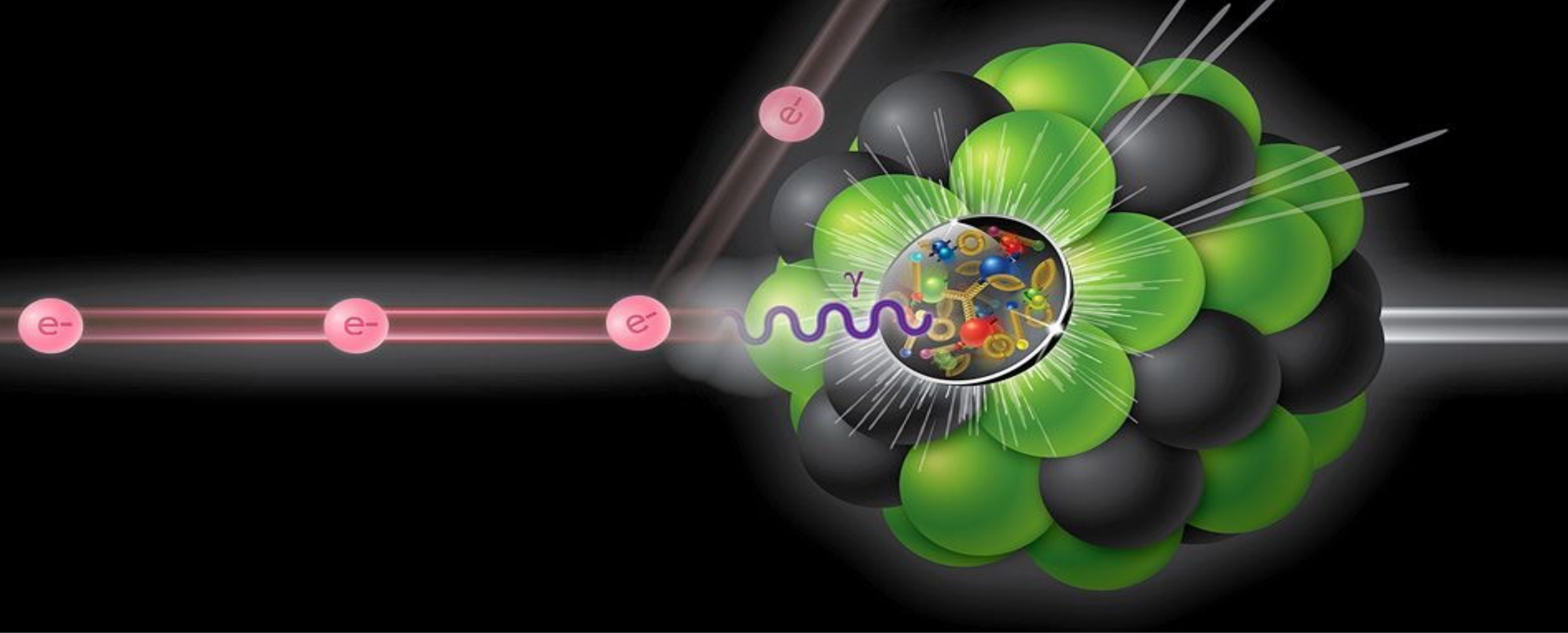




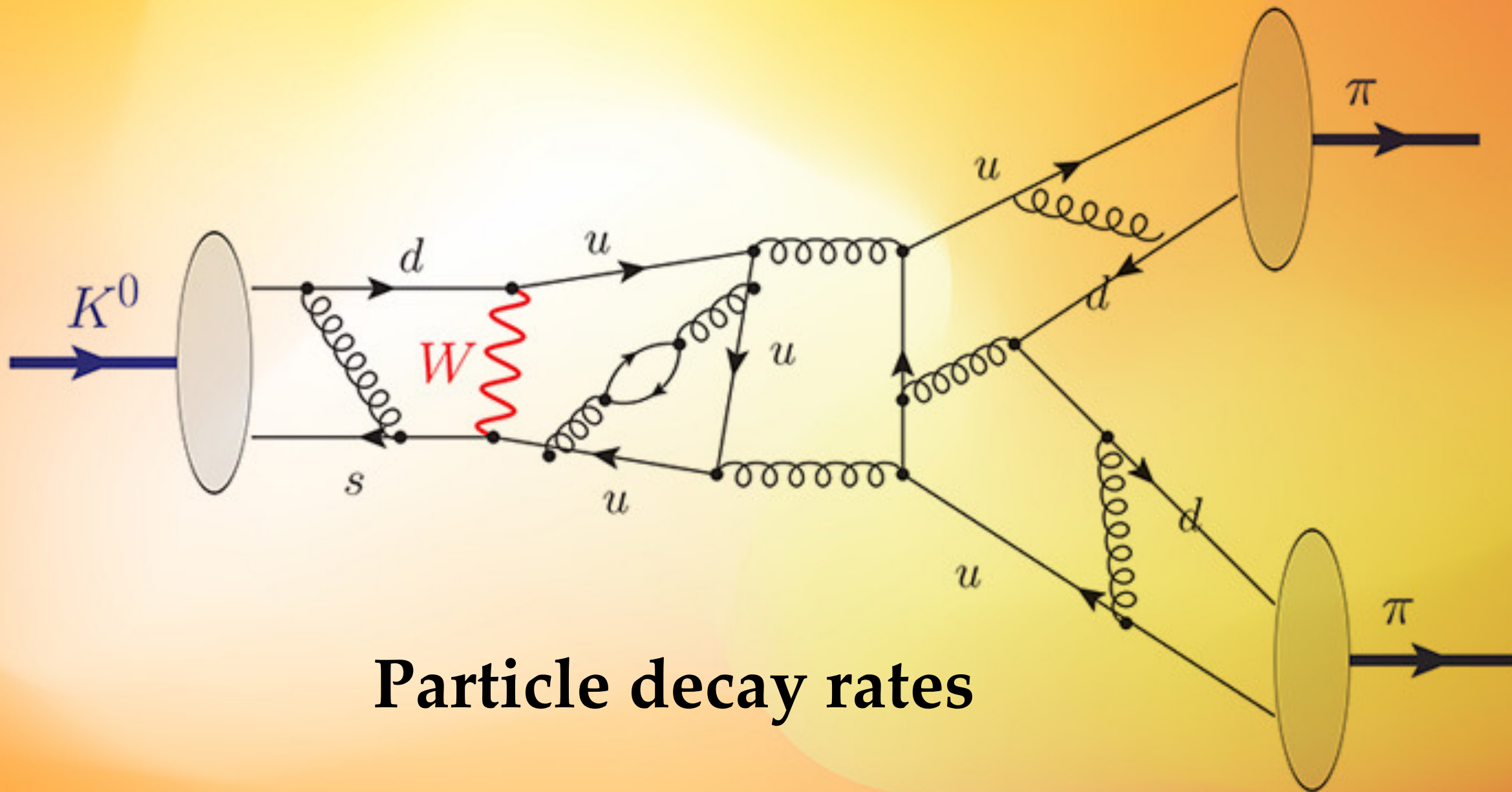
Particle Physics I

Lecture 4: Particle decay rates

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Today's learning targets

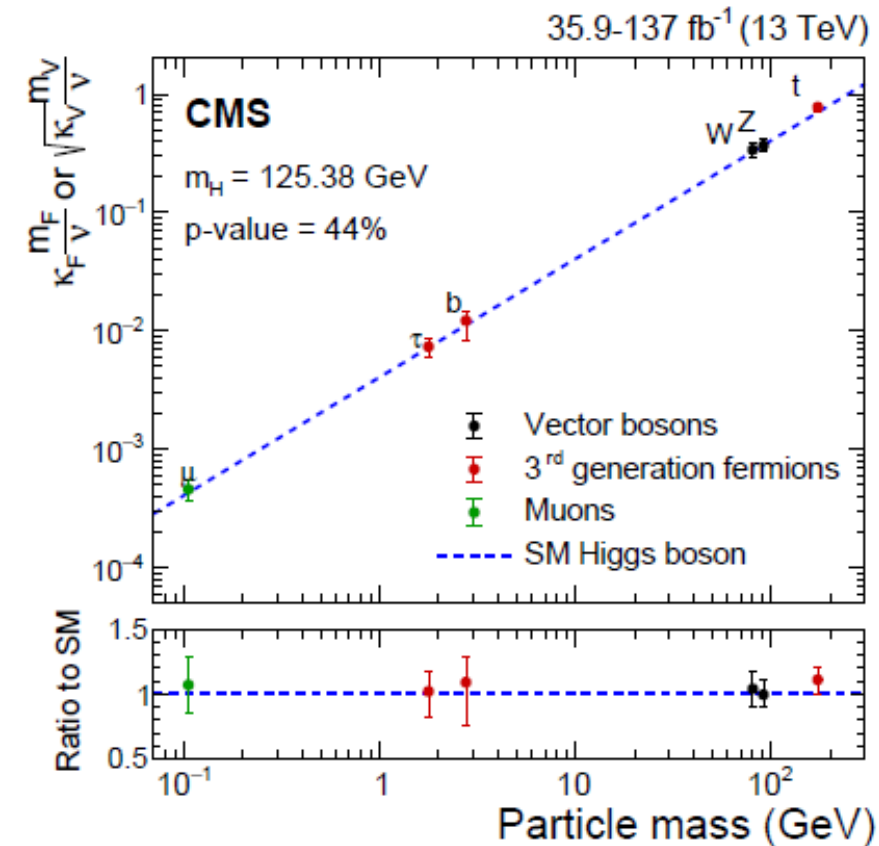
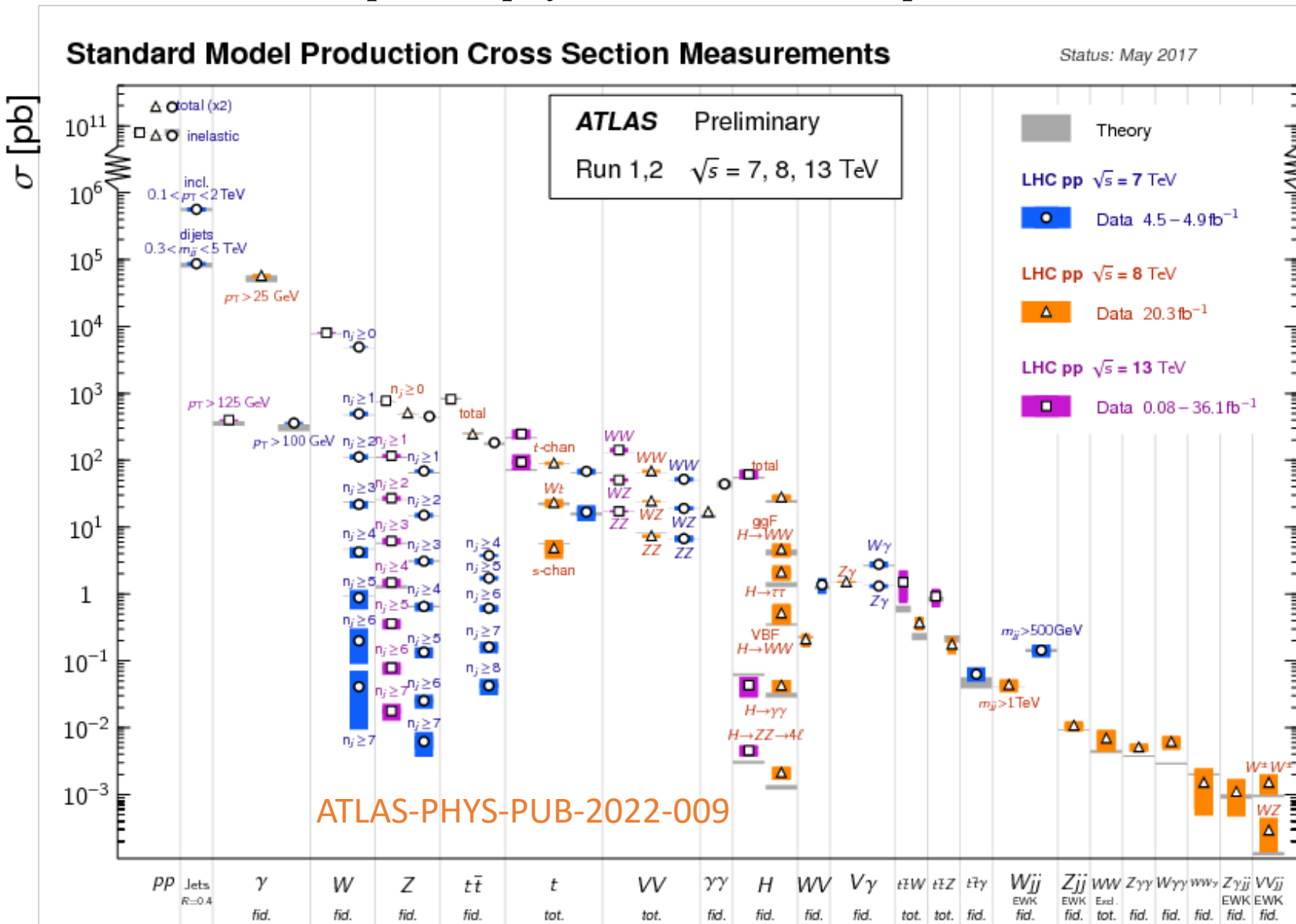
- How to compute particle decay rates
- What is Fermi's golden rule
- How to compute 2-body decay rates using Fermi's golden rule



Particle decay rates

Cross sections and decay rates

- All calculation in particle physics revolve around particle **interactions** and **decays** (transition between states)



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Cross sections and decay rates

- We can calculate transition rates using Fermi's Golden Rule (see Chapter 2.3.6 in Thomson for the derivation) :

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f) \quad (1)$$

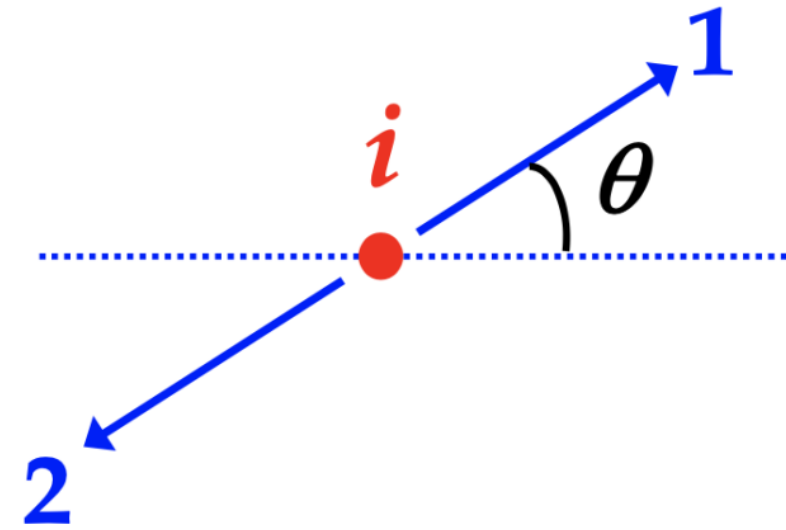
- Γ_{fi} : number of transitions per unit time from initial state $|i\rangle$ to final state $|f\rangle$ (**not Lorentz Invariant!**)
- T_{fi} : transition matrix element (ME) determined by the Hamiltonian of the interaction (j is an intermediate particle)

weak perturbation 

$$T_{fi} = \langle f | \hat{H} | i \rangle + \sum_{j \neq i} \frac{\langle f | \hat{H} | j \rangle \langle j | \hat{H} | i \rangle}{E_i - E_j} + \dots \quad (2)$$

- $\rho(E_f)$: density of final states, $\rho(E_f) = \left| \frac{dn}{dE} \right|_{E_f}$
- Decay rates depend on **matrix elements** (= fundamental particle physics) and **densities of states** (= kinematics)

Particle decay rates



- Two-body decay $i \rightarrow 1 + 2$
- The transition matrix element in first order perturbation theory is given by:

$$T_{fi} = \langle \Psi_1 \Psi_2 | \hat{H} | \Psi_i \rangle = \int_V \Psi_1^* \Psi_2^* \hat{H} \Psi_i d^3x \quad (3)$$

- Calculate the decay rate in first order perturbation theory describing the particle motion using plane-wave (Born approximation)

$$\Psi_1 = N e^{-i(\vec{p} \cdot \vec{r} - Et)} = N e^{-ip \cdot x} \quad (4)$$

N is the normalisation

Particle decay rates

- For the decay-rate computation we need to know (in a Lorentz Invariant form)
 - wave-function normalisation
 - transition matrix element from perturbation theory
 - expression for the density of states
- Let's consider wave-function normalisation first:
 - non-relativistic formulation: normalise to one particle per cube of size a

$$\int \Psi \Psi^* dV = N^2 a^3 = 1 \implies N^2 = 1/a^3 \quad (5)$$

Non-relativistic phase space

- Apply boundary conditions: $\vec{p} = \hbar \vec{k}$
- Periodic boundary conditions on the wave function:
 - $\Psi(x + a, y, z) = \Psi(x, y, z) \Rightarrow$ quantized particle momentum

$$p_x = \frac{2\pi n_x}{a}; \quad p_y = \frac{2\pi n_y}{a}; \quad p_z = \frac{2\pi n_z}{a}$$

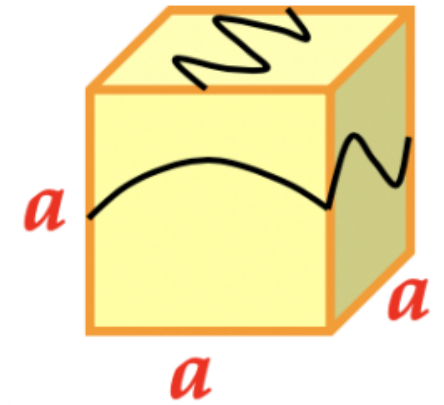
- Volume of single state in momentum space:

$$\left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$$

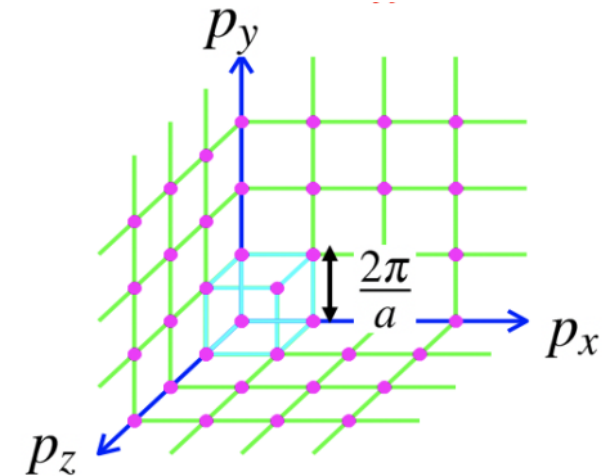
- Normalising to one particle per unit volume gives the number of states in an element

$$d^3\vec{p} = dp_x dp_y dp_z$$

$$dn = \frac{d^3\vec{p}}{(2\pi)^3/V} \times \frac{1}{V} = \frac{d^3\vec{p}}{(2\pi)^3} \quad (8)$$



(6)

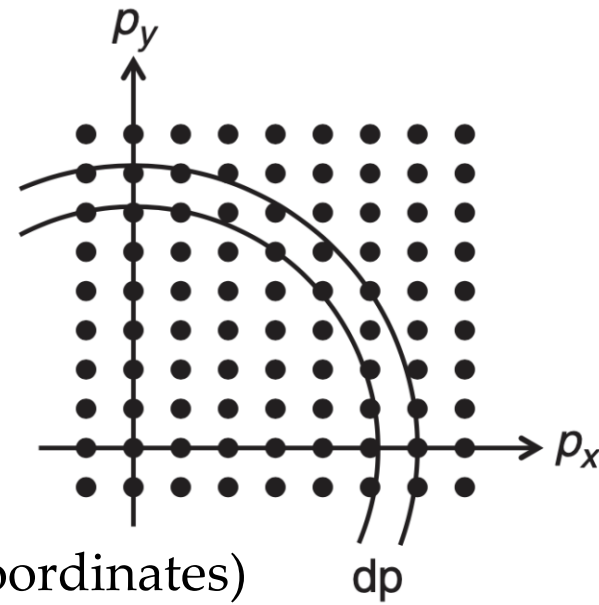


(7)

Non-relativistic phase space

- Density of states in the Golden rule: $\vec{p} = \hbar \vec{k}$

$$\rho(E_f) = \left| \frac{dn}{dE} \right|_{E_f} = \left| \frac{dn}{d|\vec{p}|} \frac{d|\vec{p}|}{dE} \right|_{E_f} \quad (9)$$



- Transformation of the elements using Eq. 8 and 9: $d^3\vec{p} = 4\pi p^2 dp$ (spherical coordinates)

$$\frac{dn}{d|\vec{p}|} = \frac{1}{(2\pi)^3} \frac{d^3\vec{p}}{d|\vec{p}|} = \frac{4\pi p^2 dp}{(2\pi)^3 dp} = \frac{4\pi p^2}{(2\pi)^3} \quad (10)$$

$$E^2 = p^2 + m^2 \Rightarrow 2E dE = 2p dp \Rightarrow \frac{dp}{dE} = \frac{E}{p} \approx \frac{1}{\beta} \quad (11)$$

$$\Rightarrow \rho(E_f) = \frac{4\pi p^2}{(2\pi)^3} \frac{1}{\beta} \quad (12)$$

- Larger final state momentum implies larger density of states (all other things being equal)
 - decays to lighter particles are preferred

The Golden rule revisited

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f) \quad (13)$$

- Rewrite the expression for density of states using a Dirac's δ –function (backup slides?)
- Transformation of the elements using Eq. 8 and 9: $d^3\vec{p} = 4\pi p^2 dp$ (spherical coordinates)

$$\rho(E_f) = \left| \frac{dn}{dE} \right|_{E_f} = \int \frac{dn}{dE} \delta(E - E_i) dE \quad \text{since } E_f = E_i \quad (14)$$

- Note: integrating over all final states energies but energy conservation now taken into account explicitly by the δ –function
- Hence the golden rule becomes an integral over all “allowed” final states of **any energy**:

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E) dn \quad (15)$$

The Golden rule revisited

- For dn in a two-body decay, we only need to consider one particle as **momentum conservation fixes the other**

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E_1 - E_2) \frac{d^3 \vec{p}_1}{(2\pi)^3} \quad (16)$$

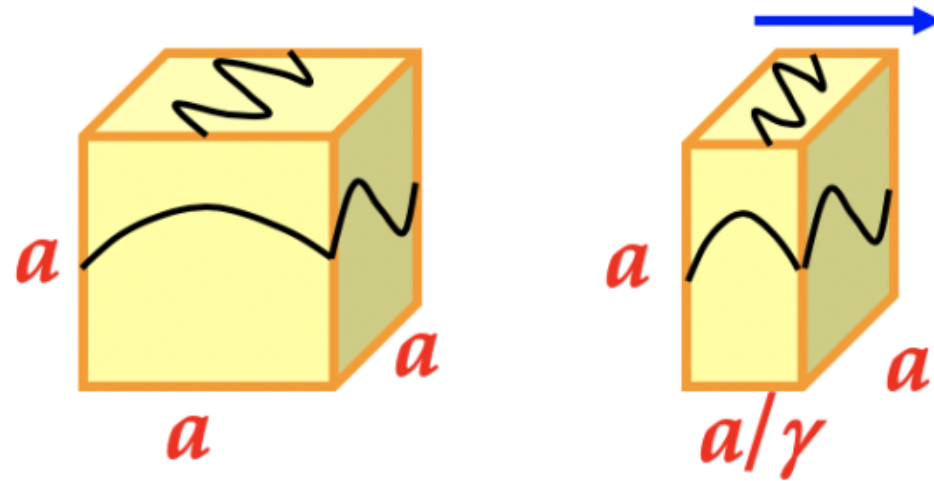
- We can also include momentum conservation explicitly by integrating over the momenta of both particles and using another δ –function

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2) \frac{d^3 \vec{p}_1}{(2\pi)^3} \frac{d^3 \vec{p}_2}{(2\pi)^3} \quad (17)$$


E conservation $\vec{p}$ conservation density of states

Lorentz invariant phase space

- In non-relativistic QM normalise to one particle/unit volume: $\int \Psi^* \Psi dV = 1$
- Considering relativistic effects: moving to different reference frame, volume **contracts by $\gamma = E/m$**



- Particle density must therefore increase by $\gamma \Rightarrow$ Lorentz invariant wave-function normalisation must be proportional to E particles per unit volume

Lorentz invariant phase space

- Usual convention: normalise to $2E$ particles per unit volume: $\int \Psi'^* \Psi' dV = 2E$
- $\Psi' = \sqrt{2E} \Psi$ is properly normalised to take into account relativistic space-time contraction
- Define Lorentz invariant matrix element, M_{fi} , in terms of the wave-functions normalized to $2E$ particles per unit volume:

$$M_{fi} = \underbrace{\langle \Psi'_1 \Psi'_2 \dots }_{\text{final state}} \underbrace{|\hat{H}| \Psi'_a \Psi'_b \dots \rangle}_{\text{initial state}} = \sqrt{2E_1 2E_2 \dots 2E_a 2E_b} \times T_{fi} \quad (18)$$

Two-body decay

$$M_{fi} = \langle \Psi'_1 \Psi'_2 | \hat{H} | \Psi_i \rangle = \sqrt{2E_1 2E_2 2E_i} \times \langle \Psi_1 \Psi_2 | \hat{H} | \Psi_i \rangle = \sqrt{2E_1 2E_2 2E_i} \times T_{fi} \quad (19)$$

- Expressing T_{fi} in terms of M_{fi} then gives

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2) \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} \quad (20)$$

- M_{fi} uses relativistically normalised wave-functions and is **Lorentz invariant**
- $\frac{d^3\vec{p}}{(2\pi)^3 2E}$ is the Lorentz invariant phase space for each final state particle
 - the factor of $2E$ arises from the wave-function normalisation

Two-body decay

- This form of Γ_{fi} is simply a rearrangement of the original equation but the integral is now frame-independent (Lorentz invariant)
- Γ_{fi} is inversely proportional to E_i , the energy of the decaying particle
 - this is an expected effect induced by time dilation
- Energy and momentum conservation are explicitly imposed by the δ –functions

Decay rate calculation

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2) \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} \quad (21)$$

- The integral is Lorentz invariant $\Rightarrow$ can be evaluated in any frame $\Rightarrow$ center-of-mass (CM) frame is the most convenient as the mother particle is at rest $\Rightarrow E_i = m_i, \vec{p}_i = 0$

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2) \delta^3(\vec{p}_1 + \vec{p}_2) \frac{d^3\vec{p}_1}{2E_1} \frac{d^3\vec{p}_2}{2E_2} \quad (22)$$

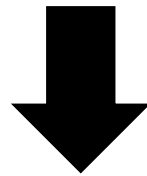
- Integrating over $\vec{p}_2$ using the delta function

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2) \frac{d^3\vec{p}_1}{4E_1 E_2} \quad (23)$$

Decay rate calculation

- The integration over $\vec{p}_2$ using the δ –function imposes $\vec{p}_2 = -\vec{p}_1$ and therefore $E_2^2 = m_2^2 + |\vec{p}_1|^2$
- We can then write $d^3\vec{p}_1 = p_1^2 dp_1 \sin\theta d\theta d\phi = p_1^2 dp_1 d\Omega$ which leads to:

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \delta\left(m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}\right) \frac{p_1^2 dp_1 d\Omega}{4E_1 E_2} \quad (24)$$



$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 g(p_1) \delta(f(p_1)) dp_1 d\Omega \quad (25)$$

Decay rate calculation

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 g(p_1) \delta(f(p_1)) dp_1 d\Omega \quad (25)$$

$$g(p_1) = \frac{p_1^2}{E_1 E_2} = \frac{p_1^2}{\sqrt{(m_1^2 + p_1^2)(m_2^2 + p_1^2)}} \quad f(p_1) = m_i - \sqrt{(m_1^2 + p_1^2)} - \sqrt{(m_2^2 + p_1^2)} \quad (26)$$

- Note that $f(p_1)$ imposes energy conservation!
- CM momenta of the two decay products is fixed by $f(p_1) = 0$ for $p_1 = -p_2 = p^*$

Decay rate calculation

- Integrating Eq. 25 and using the property of the δ –function we get

$$\int g(p_1)\delta(f(p_1))dp_1 = \frac{1}{|df/dp_1|_{p^*}} \int g(p_1)\delta(p - p^*)dp_1 = g(p^*) / \left| \frac{df}{dp_1} \right|_{p^*} \quad (27)$$

- Here, p^* is the value for which $f(p^*) = 0$

$$\frac{df}{dp_1} = -\frac{p_1}{\sqrt{m_1^2 + p_1^2}} - \frac{p_1}{\sqrt{m_1^2 + p_1^2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1 \frac{E_1 + E_2}{E_1 E_2} \quad (28)$$

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \left| \frac{E_1 E_2}{p_1(E_1 + E_2)} \frac{p_1^2}{E_1 E_2} \right|_{p_1=p^*} d\Omega = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \left| \frac{p_1}{(E_1 + E_2)} \right|_{p_1=p^*} d\Omega \quad (29)$$

- From $f(p_1) = 0$ (energy conservation) we get $m_i = E_1 + E_2$

$$\Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 E_i m_i} \int |M_{fi}|^2 d\Omega \quad (30)$$

Decay rate calculation

- In the particle's rest frame: $E_i = m_i$

$$\frac{1}{\tau} = \Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |\mathcal{M}_{fi}|^2 d\Omega \quad (30)$$

- Valid for all two-body decays – fundamental physics contained in the matrix element, additional factors arise from the phase-space integral
- p^* can be obtained from $f(p_1) = 0$

$$\begin{aligned} m_i &= \sqrt{m_1^2 + p^{*2}} + \sqrt{m_2^2 + p^{*2}} \\ \Rightarrow p^* &= \frac{1}{2m_i^2} \sqrt{[m_i^2 - (m_1 + m_2)^2][m_i^2 - (m_1 - m_2)^2]} \end{aligned} \quad (31)$$

Particle decays

- A given particle may decay to more than one decay mode
- The total decay rate per unit time Γ is the sum of the individual decay rates

$$\Gamma = \sum_j \Gamma_j \quad (32)$$

- N remaining particles after time t

$$N(t) = N(0)e^{-\Gamma t} = N(0)e^{-t/\tau} \quad , \tau = 1/\Gamma \quad (33)$$

- The branching fraction for a specific decay mode is simply given by:

$$B(j) = \frac{\Gamma_j}{\Gamma} \quad (34)$$

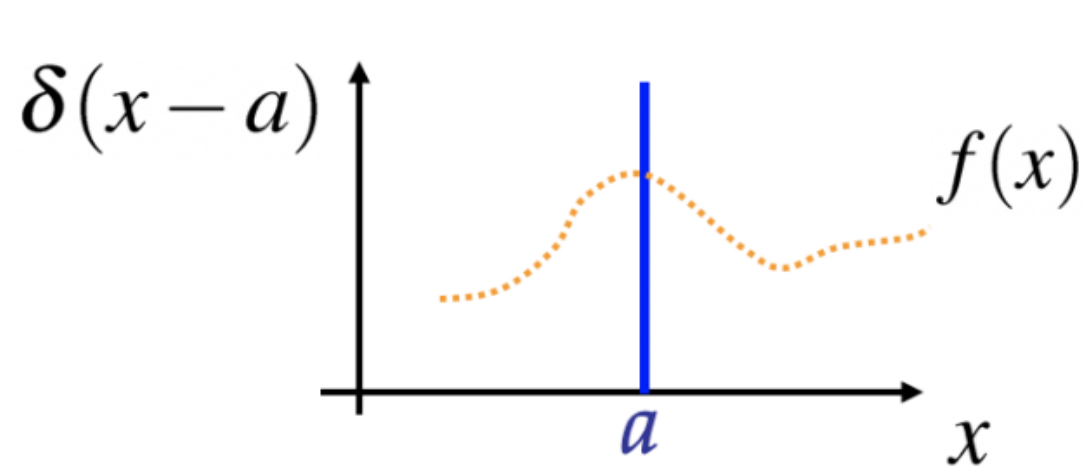
Summary of Lecture 4

Main learning outcomes

- How to compute 2-body decay rates using Fermi's golden rule
- How to deal with kinematics of particle decays
- The fundamental particle physics is in the matrix element
- The above equations are the basis for all calculations that follow

Additional slides: Dirac δ –function

- In the relativistic formulation of decay rates and cross sections we will make use of the Dirac δ function: “infinitely narrow spike of unit area”



$$\int_{-\infty}^{\infty} \delta(x - a) dx = 1$$

$$\int_{-\infty}^{\infty} f(x) \delta(x - a) dx = f(a)$$

- Any function with the above properties can represent $\delta(x)$, e.g.:

$$\delta(x) = \lim_{\sigma \rightarrow 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x^2}{2\sigma^2}\right)}$$

Infinitesimally narrow Gaussian

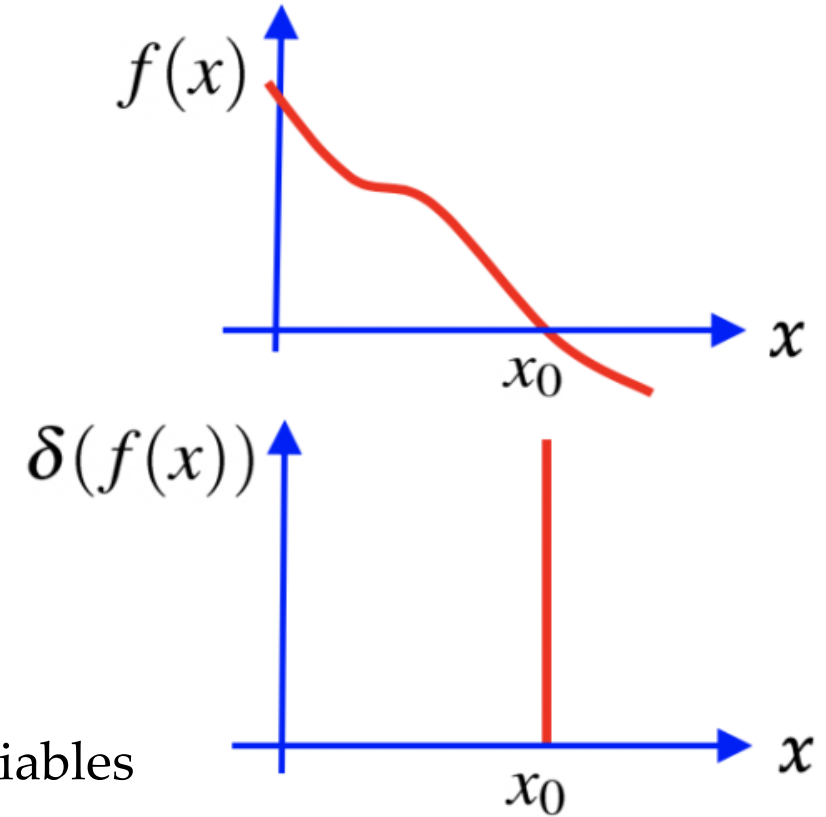
Additional slides: Dirac δ –function of a function

- An expression for the δ –function of a function $\delta(f(x))$:
 - start from the definition of a δ –function:

$$\int_{y_1}^{y_2} \delta(y) dy = \begin{cases} 1 & \text{if } y_1 < 0 < y_2 \\ 0 & \text{otherwise} \end{cases}$$

- Now express in terms of $y = f(x)$, where $f(x_0) = 0$ and change variables

$$\int_{x_1}^{x_2} \delta(f(x)) \frac{df}{dx} dx = \begin{cases} 1 & \text{if } x_1 < 0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$



Additional slides: Dirac δ –function of a function

- From the properties of a δ –function (i.e. only non-zero at x_0)

$$\left| \frac{df}{dx} \right| \int_{x_1}^{x_2} \delta(f(x)) dx = \begin{cases} 1 & \text{if } x_1 < 0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

- Rearranging and expressing RHS as a δ –function

$$\int_{x_1}^{x_2} \delta(f(x)) dx = \frac{1}{\left| \frac{df}{dx} \right|_{x_0}} \int_{x_1}^{x_2} \delta(x - x_0) dx \Rightarrow \delta(f(x)) = \left| \frac{df}{dx} \right|_{x_0}^{-1} \delta(x - x_0)$$